

**YESHIVA UNIVERSITY
GRADUATE PROGRAM IN MATHEMATICAL SCIENCES
TOPICS FOR THE PHD QUALIFYING EXAMINATION**

The qualifying examination in mathematical sciences covers three areas:

- (I) Real Analysis
- (II) Complex Analysis
- (III) Research Area

For the first two areas, a list of topics is below. Also, a list of sample exercises for the two areas is provided. The actual exercises asked on the exam will be different from the sample exercises; being able to solve the sample exercises is not sufficient for the exam preparation.

The third exam area pertains to the research subject that the student intends

7. Riemann-Lebesgue Theorem (outline). A function $f : [a; b] \rightarrow \mathbb{R}$ is Riemann integrable if and only if it is bounded and its set of discontinuity points is of measure zero.
8. Sequences of functions (uniform convergence, properties, equi-continuity for a family of functions, Ascoli-Arzelà's theorem).

Complex Analysis:

1. If f is complex differentiable at z then the Cauchy-Riemann equations are satisfied at z .
2. If the partial derivatives of u and v exist and are continuous at $(x; y)$ and the Cauchy-Riemann equations are satisfied then $f(z) = u(x; y) + iv(x; y)$ is complex differentiable at $z = x + iy$.
3. If $f'(z) = 0$ in a region D then f is constant on D .
4. If $|f(z)| < M$ on a curve C then $\int_C f(z) dz < ML$ where L is the length of the curve.
5. The following statements are equivalent:
 - (i) f has an antiderivative F ;
 - (ii) $\int_{z_1}^{z_2} f(z) dz = F(z_2) - F(z_1)$;
 - (iii) If C is a closed curve then $\int_C f(z) dz = 0$.
6. Cauchy-Goursat Theorem (outline). If f is analytic on and inside a simple closed curve C then $\int_C f(z) dz = 0$.
7. If f is analytic in the region between closed curves C_2 and C_1 with C_1 inside C_2 then

$$\int_{C_1} f(z) dz = \int_{C_2} f(z) dz:$$
8. The Cauchy Integral Formula.
9. A bounded entire function is constant.
10. If f is analytic on annulus, it equals its Laurent series (outline).
11. Cauchy Residue Theorem.

Sample Exercises:

1. Let

$$f(x) = \begin{cases} \sin \frac{1}{x} & ; \text{ for } x \neq 0; \\ & ; \text{ for } x = 0, \end{cases}$$

where $x \in [-1; 1]$.

(a) Is f continuous?

(b) Does f have the intermediate value property?

2. Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & ; \text{ for } x \neq 0; \\ 0 & ; \text{ for } x = 0, \end{cases}$$

(a) Show that f is differentiable everywhere.

(b) Is f' continuous?

3. Given that f is a quadratic polynomial

$$f(x) = Kx^2 +$$

6. On uniform convergence.

(a) Show that if a sequence $\{f_n\}$ of continuous functions converges uniformly on a domain $D \subset \mathbb{R}$ to a function f , then the limit f is also continuous on D .

(b) Let $\{f_n\}$ be a sequence of continuously differentiable functions such that $\{f_n\}$ and $\{f_n'\}$ converge uniformly on a domain D to the limiting functions f and g , respectively. Show that for every x in the interior of D ,

$$g(x) = \lim_{n \rightarrow \infty} f_n'(x) = \left(\lim_{n \rightarrow \infty} f_n(x) \right)' = f'(x):$$

7. State Ascoli-Arzelà's Theorem and outline its proof.

8. The sequence of continuous functions $\{f_n\} : [0; 2\pi] \rightarrow \mathbb{R}, n \in \mathbb{N}$ with f_n given by $f_n(x) = \sin(nx)$ is uniformly bounded, but not equicontinuous. Give an intuitive reason why such a sequence is not equicontinuous, then give a rigorous proof.

9. Compute the following integral limit

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} \frac{n \sin(x/n)}{x(x^2 + 1)} dx$$

10. Consider the function

$$f(z) = \begin{cases} \frac{z^2}{z} & \text{if } z \neq 0; \\ 0 & \text{if } z = 0; \end{cases}$$

Is this function differentiable at $z = 0$? Is it continuous at

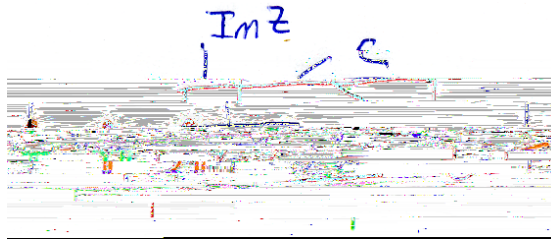


Figure 1: Contour C

15. Show that the only conformal maps from the complex plane onto itself are the non-constant linear maps, i.e. maps of the form $f(z) = az + b$, $a \neq 0$.
16. Let f be a *doubly periodic function*, that is, there are two complex numbers w_1, w_2 with $w_1, w_2 \notin \mathbb{R}$ so that for any $z \in \mathbb{C}$, $f(z) = f(z + w_1) = f(z + w_2)$. Let us also assume that f is meromorphic.
 - (a) Show that if f is an entire function, then it has to be constant.
 - (b) Let γ be the boundary of the parallelogram with vertices $0, w_1, w_2, w_1 + w_2$, oriented counterclockwise. Show that if f is analytic on γ , then $\int_{\gamma} f(z) dz = 0$.
 - (c) Assuming that f is analytic on γ and has exactly one singularity inside γ , show that the residue at this singularity is necessarily zero.

Bibliography:

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